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CO-ORDINATE TRANSFORMATIONS FOR SECOND ORDER SYSTEMS. PART II: ELEMENTARY STRUCTURE-PRESERVING TRANSFORMATIONS

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It has been shown in a previous paper that there is a real-valued transformation from the general N-degree-of-freedom second order system to a second order system characterized by diagonal matrices. An immediate extension of this fact is that for any second order system, there is a set of real-valued transformations (the *structure-preserving* transformations) which transform this system to a different second order system having identical characteristic behaviour. There are several possible reasons why it may be very useful to achieve a particular structure in the transformed system. It is obvious that a diagonal structure is extremely useful and a method has been devised for determining the diagonalizing transformation from the solution of the usual (complex) eigenvalue–eigenvector problem.

This paper begins by outlining the usefulness of some other structures. Then it defines a class of *elementary structure-preserving co-ordinate transformations* that transform from one N-degree-of-freedom second order system to another. The term *elementary* is applied because any one of these transformations is the minimum-rank modification of the identity transformation. The changes occurring in the system matrices as a result of the application of one such elementary transformation transpire to be very simple in form, they are low rank, and they can be computed very efficiently.

This paper provides the fundamental tools to enable the design of structure-preserving co-ordinate transformations which transform a second order system originally characterized by three general matrices in stages into a mathematically similar second order system characterized by three diagonal matrices. The procedure by which the individual elementary transformations are obtained is still under development and it is not discussed in this paper. However, an illustration is given of a five-degree-of-freedom self-adjoint system being transformed into tridiagonal form.

1. INTRODUCTION

In the analysis of the vibrations of undamped or classically damped structures, the matter of executing co-ordinate transformations is well established and understood. In every such case, there is one co-ordinate set—the principal or modal co-ordinates—under which the

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system matrices become diagonal. For generally damped structures, it is commonly accepted that there is no co-ordinate transformation which can simultaneously diagonalize the three system matrices. This resignation is wrong. In fact there is a co-ordinate transformation which simultaneously diagonalizes the three system matrices but this transformation is more general than the transformation used for first order or classically damped systems [1]. In the case of classically damped structures, the ability to find a diagonalizing transformation is only one of a large set of capabilities made available to the vibrations engineer through the existence of co-ordinate transformations. This paper aims to provide the basis for the extension of this set of capabilities to systems having general viscous damping through the provision of elementary structure-preserving transformations for such systems.

The second order system of interest is that described by matrices $\mathbf{K}$, $\mathbf{D}$ and $\mathbf{M}$ and having $\mathbf{q}$ as its vector of displacement co-ordinates and $\mathbf{Q}$ as its vector of forces,

$$\mathbf{Kq} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}. \tag{1}$$

Although it is common that the system matrices are symmetric, this is not always the case and therefore it is possible that different transformations are applied to the left- and right-hand sides of the system. These will be distinguished using the subscripts L and R respectively.

In this paper, all of the transformations applied will all be *square* in the sense that the transformed system matrices will have the same dimensions as the matrices of the original system. There is a major role for *model-reducing* transformations and this paper prepares much relevant ground for such transformations insofar as every model-reducing transformation can be understood to be a square transformation followed by the simple discarding of some system degrees of freedom. Any further treatment of elementary model-reducing transformations is deferred to another paper.

Any eight real $(N \times N)$ matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ define a general co-ordinate transformation for a second order system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ [1] but this transformation is *structure preserving* if and only if

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}' \\ \mathbf{K}' & \mathbf{D}' \end{bmatrix}, \tag{2}$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}' & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}' \end{bmatrix}, \tag{3}$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{D}' & \mathbf{M}' \\ \mathbf{M}' & \mathbf{0} \end{bmatrix}, \tag{4}$$

where $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ are the system matrices of a new second order system. The identity transformation $(\mathbf{W}_L = \mathbf{I} = \mathbf{Z}_L, \mathbf{X}_L = \mathbf{0} = \mathbf{Y}_L, \mathbf{W}_R = \mathbf{I} = \mathbf{Z}_R, \mathbf{X}_R = \mathbf{0} = \mathbf{Y}_R)$ clearly results in $\mathbf{K}' = \mathbf{K}$, $\mathbf{D}' = \mathbf{D}$, $\mathbf{M}' = \mathbf{M}$.

The concern of this paper is to provide elementary structure-preserving co-ordinate transformations which are low-rank modifications of the identity transformation. With these, it will be possible to design numerically stable co-ordinate transformations such that after each one, system $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ has “more structure” than $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$.

2. STRUCTURE IN THE SYSTEM MATRICES

System matrices {$\mathbf{K}$, $\mathbf{D}$, $\mathbf{M}$} may be derived for any model of a second order system and they are often densely populated. In these cases, the direct computation of exact frequency response at any one forcing frequency, ω_f, involves the solution of N coupled simultaneous equations in N complex unknowns. In lumped-mass system models, the mass matrix, $\mathbf{M}$, may be diagonal. This fact is not useful in the direct computation of exact frequency response since at every finite frequency, ω_f, the complex ($N \times N$) dynamic stiffness matrix may still be densely populated. Indeed, even if any two of the system matrices are diagonal, there is evidently no advantage in the direct computation of exact frequency response. To gain substantial advantage, it is necessary that the complex dynamic stiffness matrix for the system has some particular structure for all frequencies, ω_f.

If all three system matrices {$\mathbf{K}$, $\mathbf{D}$, $\mathbf{M}$} are diagonal, the direct computation of exact frequency response at any one forcing frequency, ω_f, involves only the solution of N decoupled equations, each involving only one complex unknown and the advantage afforded by the diagonal structure is very large for large N—being in the order of N^2.

In Part I [1], it was shown that given (almost) any original second order system {$\mathbf{K}$, $\mathbf{D}$, $\mathbf{M}$}, a real-valued co-ordinate transformation exists to transform this into diagonal form. This transformation can be derived readily from the usual (complex) modal information. The fact that solving for the exact frequency response of a system represented by diagonal {$\mathbf{K}$, $\mathbf{D}$, $\mathbf{M}$} is very computationally efficient is a direct reflection of the fact that the exact frequency response of any second order system can be determined very efficiently using its complex modal data. The role of diagonal structure in the system matrices is thus above question.

Although the diagonal structure is unquestionably the ultimate one in terms of computational efficiency, there are other matrix structures of substantial practical interest. Excluding "sparse" matrices (on the grounds that there is no obvious role for co-ordinate transformations in connection with these) these other matrix structures of interest include:

(a) tridiagonal,
(b) banded,
(c) banded with a bulge,
(d) bordered diagonal,
(e) bordered tridiagonal,
(f) bordered banded.

It is neither necessary nor appropriate to provide formal definitions for these here. Since *diagonal* and *tridiagonal* forms are special cases of *banded* matrices with half-bandwidths of 0 and 1, respectively, only classes (b), (c) and (f) require definition and a pictorial representation of these structures is sufficient for present purposes (see Figure 1).

Systems described by banded matrices have been shown to have practical importance in vibration analysis—for example in component mode synthesis for structures having rigid connections [2] and in the context of determining the resonances of periodic structures [3]. Systems described by bordered–banded (bordered-diagonal) matrices appear in the context of performing structural modifications to locate zeros deliberately [4]. The reasons for mentioning the banded-with-a-bulge structure emerge subsequently.

Systems whose dynamic stiffness matrix is either banded or bordered–banded may still afford very substantial computational advantages in the direct computation of exact frequency response over systems whose dynamic stiffness matrix is not structured. This is true whenever the width of the band (and border) is very small compared with the overall dimension of the matrix.

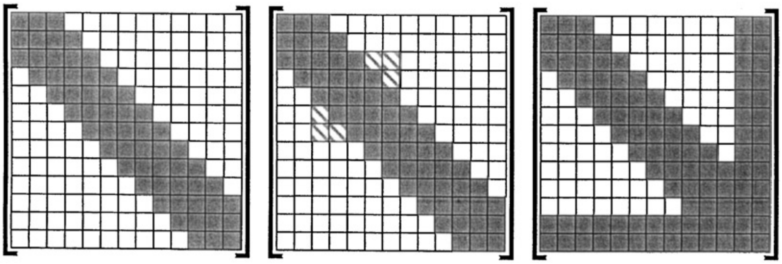

Figure 1. Banded, banded-with-a-bulge, and bordered–banded matrix structures.

The discussion on structure has focused up to now on frequency response. It could be extended to include the role of matrix structure in time-domain response computation and in the computation of characteristic roots where it is equally important but this extension would detract from the central thrust of the paper. The purpose of this section is to motivate the remainder of the paper by supporting this assertion

Transforming a given second order system $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$ to a new form $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ in which the dynamic stiffness matrix has a banded or bordered–banded structure is a worthwhile action provided that the transformation itself can be done efficiently.

The objective of the paper is therefore to expose classes of elementary structure-preserving co-ordinate transformations that will provide the means for this transformation.

One feature which makes the direct transformation to banded form particularly attractive is the possibility that the diagonalizing transformation might potentially be found from the banded form using a variation of the concept of *bulge chasing*. The banded-with-a-bulge structure shown in Figure 1 shows some out-of-band non-zeros part way down the matrix. By performing a suitable structure-preserving transformation, it is possible that the bulge can be shifted one step further down the matrix. Subsequent steps can shift this bulge repeatedly downwards until finally it disappears at the bottom right corner. There are numerous algorithms [5] for the use of bulge-chasing methods in the standard eigenvalue problem for both upper-Hessenberg and tridiagonal matrices. Extensions of these algorithms can be envisaged which utilize the elementary structure-preserving transformations.

3. ELEMENTARY STRUCTURE-PRESERVING CO-ORDINATE TRANSFORMATIONS FOR SECOND ORDER SYSTEMS

Equations (2)–(4) require that nine different matrix identities be satisfied if the transformation is *structure preserving*. Two of these are connected with forcing all occurrences of $\mathbf{K}'$ in equations (2) and (3) to be identical:

$$\begin{aligned} \mathbf{W}_L^{\mathrm{T}}\mathbf{K}\mathbf{Z}_R + \mathbf{Y}_L^{\mathrm{T}}\mathbf{K}\mathbf{X}_R + \mathbf{Y}_L^{\mathrm{T}}\mathbf{D}\mathbf{Z}_R &= \mathbf{W}_L^{\mathrm{T}}\mathbf{K}\mathbf{W}_R - \mathbf{Y}_L^{\mathrm{T}}\mathbf{M}\mathbf{Y}_R, \\ \mathbf{X}_L^{\mathrm{T}}\mathbf{K}\mathbf{Y}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{K}\mathbf{W}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{D}\mathbf{Y}_R &= \mathbf{W}_L^{\mathrm{T}}\mathbf{K}\mathbf{W}_R - \mathbf{Y}_L^{\mathrm{T}}\mathbf{M}\mathbf{Y}_R. \end{aligned} \tag{5}$$

A further two equations enforce the equality between all occurrences of $\mathbf{M}'$ in equations (3) and (4).

$$\begin{aligned} \mathbf{W}_L^{\mathrm{T}}\mathbf{D}\mathbf{X}_R + \mathbf{W}_L^{\mathrm{T}}\mathbf{M}\mathbf{Z}_R + \mathbf{Y}_L^{\mathrm{T}}\mathbf{M}\mathbf{X}_R &= -\mathbf{X}_L^{\mathrm{T}}\mathbf{K}\mathbf{X}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{M}\mathbf{Z}_R, \\ \mathbf{X}_L^{\mathrm{T}}\mathbf{D}\mathbf{W}_R + \mathbf{X}_L^{\mathrm{T}}\mathbf{M}\mathbf{Y}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{M}\mathbf{W}_R &= -\mathbf{X}_L^{\mathrm{T}}\mathbf{K}\mathbf{X}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{M}\mathbf{Z}_R. \end{aligned} \tag{6}$$

A separate equation again acts to ensure that the occurrences of $\mathbf{D}'$ in equations (2) and (4) are the same.

$$\mathbf{X}_L^{\mathrm{T}}\mathbf{K}\mathbf{Z}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{K}\mathbf{X}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{D}\mathbf{Z}_R = \mathbf{W}_L^{\mathrm{T}}\mathbf{D}\mathbf{W}_R + \mathbf{W}_L^{\mathrm{T}}\mathbf{M}\mathbf{Y}_R + \mathbf{Y}_L^{\mathrm{T}}\mathbf{M}\mathbf{W}_R. \tag{7}$$

Finally, four equations assert the zero blocks in equations (2)–(4). Equation (2) produces one of these equations,

$$\mathbf{W}_L^{\mathrm{T}}\mathbf{K}\mathbf{Y}_R + \mathbf{Y}_L^{\mathrm{T}}\mathbf{K}\mathbf{W}_R + \mathbf{Y}_L^{\mathrm{T}}\mathbf{D}\mathbf{Y}_R = \mathbf{0}. \tag{8}$$

Equation (3) produces these two:

$$\mathbf{W}_L^{\mathrm{T}}\mathbf{K}\mathbf{X}_R - \mathbf{Y}_L^{\mathrm{T}}\mathbf{M}\mathbf{Z}_R = \mathbf{0}, \quad \mathbf{X}_L^{\mathrm{T}}\mathbf{K}\mathbf{W}_R - \mathbf{Z}_L^{\mathrm{T}}\mathbf{M}\mathbf{Y}_R = \mathbf{0} \tag{9}$$

and equation (4) produces

$$\mathbf{X}_L^{\mathrm{T}}\mathbf{D}\mathbf{X}_R + \mathbf{X}_L^{\mathrm{T}}\mathbf{M}\mathbf{Z}_R + \mathbf{Z}_L^{\mathrm{T}}\mathbf{M}\mathbf{X}_R = \mathbf{0}. \tag{10}$$

These equations are not all independent. It was noted previously [1] that equation (4) is automatic given equations (2) and (3). Hence it is acceptable to adopt only those criteria for structure preservation arising completely from equations (2) and (3) combined. These are equations (5), (8) and (9).

From this point onwards, attention is restricted to the case where $\{\mathbf{W}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{Z}_R\}$ are each unit-rank modifications of the identity matrix and $\{\mathbf{X}_L, \mathbf{Y}_L, \mathbf{X}_R, \mathbf{Y}_R\}$ are unit-rank matrices. Under these restrictions, there must exist some eight N vectors $\{\mathbf{a}_L, \mathbf{b}_L, \mathbf{c}_L, \mathbf{d}_L, \mathbf{e}_L, \mathbf{f}_L, \mathbf{g}_L, \mathbf{h}_L\}$ defining the left transformation according to

$$\begin{aligned} \mathbf{W}_L &= (\mathbf{I} + \mathbf{a}_L\mathbf{b}_L^{\mathrm{T}}), & \mathbf{X}_L &= (\mathbf{c}_L\mathbf{d}_L^{\mathrm{T}}), \\ \mathbf{Y}_L &= (\mathbf{e}_L\mathbf{f}_L^{\mathrm{T}}), & \mathbf{Z}_L &= (\mathbf{I} + \mathbf{g}_L\mathbf{h}_L^{\mathrm{T}}) \end{aligned} \tag{11}$$

and a further eight N vectors $\{\mathbf{a}_R, \mathbf{b}_R, \mathbf{c}_R, \mathbf{d}_R, \mathbf{e}_R, \mathbf{f}_R, \mathbf{g}_R, \mathbf{h}_R\}$ defining the *right* transformation according to

$$\begin{aligned} \mathbf{W}_R &= (\mathbf{I} + \mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}), & \mathbf{X}_R &= (\mathbf{c}_R\mathbf{d}_R^{\mathrm{T}}), \\ \mathbf{Y}_R &= (\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}}), & \mathbf{Z}_R &= (\mathbf{I} + \mathbf{g}_R\mathbf{h}_R^{\mathrm{T}}). \end{aligned} \tag{12}$$

Substituting equations (11) and (12) into equations (5), (8) and (9) produces five different conditions on the 16 vectors. Appendix A develops these conditions into a simple form. It shows that there are two distinct classes of elementary structure preserving transformations according to equations (11) and (12).

3.1. CLASS-1 ELEMENTARY STRUCTURE-PRESERVING TRANSFORMATIONS

Any set of vectors $\{\mathbf{a}_L, \mathbf{b}_L, \mathbf{c}_L, \mathbf{d}_L, \mathbf{e}_L, \mathbf{f}_L, \mathbf{g}_L, \mathbf{h}_L\}$ and $\{\mathbf{a}_R, \mathbf{b}_R, \mathbf{c}_R, \mathbf{d}_R, \mathbf{e}_R, \mathbf{f}_R, \mathbf{g}_R, \mathbf{h}_R\}$ obeying

$$\begin{aligned} &\mathbf{g}_L = \mathbf{a}_L, \quad \mathbf{h}_L = \mathbf{b}_L, \quad \mathbf{c}_L = \mathbf{d}_L = \mathbf{e}_L = \mathbf{f}_L = \mathbf{0}, \\ &\mathbf{g}_R = \mathbf{a}_R, \quad \mathbf{h}_R = \mathbf{b}_R, \quad \mathbf{c}_R = \mathbf{d}_R = \mathbf{e}_R = \mathbf{f}_R = \mathbf{0} \end{aligned} \tag{13}$$

defines one of these transformations (according to equations (11) and (12)) and preserves structure. Evidently, only four N vectors may be chosen independently here ($\mathbf{a}_L$, $\mathbf{b}_L$, $\mathbf{a}_R$, $\mathbf{b}_R$), and there are two degrees of redundancy in this choice. For improved clarity in the remainder of this paper, all Class-1 elementary structure-preserving transformations are represented using vectors $\{\mathbf{m}_L, \mathbf{n}_L, \mathbf{m}_R, \mathbf{n}_R\}$ in place of $\{\mathbf{a}_L, \mathbf{b}_L, \mathbf{a}_R, \mathbf{b}_R\}$, respectively, and these transformations can be described (more succinctly than equations (11) and (12)) by

$$\begin{aligned} \mathbf{W}_L &= (\mathbf{I} + \mathbf{m}_L\mathbf{n}_L^{\mathrm{T}}) = \mathbf{Z}_L, \quad \mathbf{X}_L = \mathbf{0} = \mathbf{Y}_L, \\ \mathbf{W}_R &= (\mathbf{I} + \mathbf{m}_R\mathbf{n}_R^{\mathrm{T}}) = \mathbf{Z}_R, \quad \mathbf{X}_R = \mathbf{0} = \mathbf{Y}_R. \end{aligned} \tag{14}$$

3.2. CLASS-2 ELEMENTARY STRUCTURE-PRESERVING TRANSFORMATIONS

Any set of vectors $\{\mathbf{a}_L, \mathbf{b}_L, \mathbf{c}_L, \mathbf{d}_L, \mathbf{e}_L, \mathbf{f}_L, \mathbf{g}_L, \mathbf{h}_L\}$ and $\{\mathbf{a}_R, \mathbf{b}_R, \mathbf{c}_R, \mathbf{d}_R, \mathbf{e}_R, \mathbf{f}_R, \mathbf{g}_R, \mathbf{h}_R\}$ obeying

$$\mathbf{g}_L = \mathbf{e}_L = \mathbf{c}_L = \mathbf{a}_L, \quad \mathbf{g}_R = \mathbf{e}_R = \mathbf{c}_R = \mathbf{a}_R \tag{15}$$

and satisfying

$$\begin{aligned} \mathbf{K}^{\mathrm{T}}\mathbf{a}_L + \mathbf{b}_R x_K + \mathbf{f}_R x_D = \mathbf{0}, &\quad \mathbf{K}\mathbf{a}_R + \mathbf{b}_L x_K + \mathbf{f}_L x_D = \mathbf{0}, \\ \mathbf{D}^{\mathrm{T}}\mathbf{a}_L + \mathbf{d}_R x_K + (\mathbf{b}_R + \mathbf{h}_R) x_D + \mathbf{f}_R x_M = \mathbf{0}, &\quad \mathbf{D}\mathbf{a}_R + \mathbf{d}_L x_K + (\mathbf{b}_L + \mathbf{h}_L) x_D + \mathbf{f}_L x_M = \mathbf{0}, \\ \mathbf{M}^{\mathrm{T}}\mathbf{a}_L + \mathbf{d}_R x_D + \mathbf{h}_R x_M = \mathbf{0}, &\quad \mathbf{M}\mathbf{a}_R + \mathbf{d}_L x_D + \mathbf{h}_L x_M = \mathbf{0}, \end{aligned} \tag{16}$$

where the scalars $\{x_K, x_D, x_M\}$ are defined by

$$x_K := (\mathbf{a}_L^{\mathrm{T}}\mathbf{K}\mathbf{a}_R), \quad x_D := \tfrac{1}{2}(\mathbf{a}_L^{\mathrm{T}}\mathbf{D}\mathbf{a}_R), \quad x_M := (\mathbf{a}_L^{\mathrm{T}}\mathbf{M}\mathbf{a}_R) \tag{17}$$

defines one of these transformations (according to equations (11) and (12)) and preserves structure. Once again, four different N vectors may be chosen independently and there are two degrees of redundancy in this choice. Provided that vectors $\mathbf{a}_R$ and $\mathbf{a}_L$ are chosen directly, equations (16) are linear with respect to the other vectors. Selecting $(\mathbf{b}_L - \mathbf{h}_L)$ and $(\mathbf{b}_R - \mathbf{h}_R)$ in addition to $\mathbf{a}_R$ and $\mathbf{a}_L$ is attractive (where an arbitrary Class-2 transformation is desired) for symmetry reasons. Note that $(\mathbf{b}_L + \mathbf{h}_L)$ and $(\mathbf{b}_R + \mathbf{h}_R)$ are fully determined once $\mathbf{a}_R$ and $\mathbf{a}_L$ are known since Appendix A shows that for these transformations

$$\begin{aligned} (\mathbf{b}_R + \mathbf{h}_R) &= -\frac{(\mathbf{M}^{\mathrm{T}}\mathbf{a}_L x_K - \mathbf{D}^{\mathrm{T}}\mathbf{a}_L x_D + \mathbf{K}^{\mathrm{T}}\mathbf{a}_L x_M)}{(x_K x_M - x_D^2)}, \\ (\mathbf{b}_L + \mathbf{h}_L) &= -\frac{(\mathbf{M}\mathbf{a}_R x_K - \mathbf{D}\mathbf{a}_R x_D + \mathbf{K}\mathbf{a}_R x_M)}{(x_K x_M - x_D^2)}. \end{aligned} \tag{18}$$

Because of equation (15), no further reference is made to vectors $\{\mathbf{c}_L, \mathbf{e}_L, \mathbf{g}_L\}$ since they are represented perfectly by $\mathbf{a}_L$. Similarly for $\{\mathbf{c}_R, \mathbf{e}_R, \mathbf{g}_R\}$ since they are represented perfectly by $\mathbf{a}_R$. Hence, an elementary transformation of this type is considered to be represented by only 10 vectors, $\{\mathbf{a}_L, \mathbf{b}_L, \mathbf{d}_L, \mathbf{f}_L, \mathbf{h}_L\}$ representing the left transformation and $\{\mathbf{a}_R, \mathbf{b}_R, \mathbf{d}_R, \mathbf{f}_R, \mathbf{h}_R\}$ representing the right transformation.

4. THE CHANGES BROUGHT ABOUT BY THE ELEMENTARY TRANSFORMATIONS

Two distinct classes of elementary structure-preserving transformation have now been defined. Because each of these transformations are low-rank modifications of the identity transformation, the changes in the system matrices brought about by the transformations will also be low rank. The Class-1 and Class-2 elementary transformations are discussed separately. In both cases, matrices $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ will represent the matrices of the new system obtained after the transformation.

4.1. CLASS-1 ELEMENTARY STRUCTURE-PRESERVING TRANSFORMATIONS

Equations (14) describe the transformation in terms of the four vectors $\{\mathbf{m}_L, \mathbf{n}_L, \mathbf{m}_R, \mathbf{n}_R\}$ and equations (2)–(4) show how this transformation applies to the original second order system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, to produce the new system $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$. In the case of Class-1 elementary transformations, the relationship between the original system and the transformed one can be written much more simply as

$$\begin{aligned} &\mathbf{K}' = (\mathbf{I} + \mathbf{n}_L\mathbf{m}_L^{\mathrm{T}})\mathbf{K}(\mathbf{I} + \mathbf{m}_R\mathbf{n}_R^{\mathrm{T}}), \quad \mathbf{D}' = (\mathbf{I} + \mathbf{n}_L\mathbf{m}_L^{\mathrm{T}})\mathbf{D}(\mathbf{I} + \mathbf{m}_R\mathbf{n}_R^{\mathrm{T}}), \\ &\mathbf{M}' = (\mathbf{I} + \mathbf{n}_L\mathbf{m}_L^{\mathrm{T}})\mathbf{M}(\mathbf{I} + \mathbf{m}_R\mathbf{n}_R^{\mathrm{T}}). \end{aligned} \tag{19}$$

The changes occurring in the system matrices as a result of a single Class-1 elementary transformation are then easily derived as

$$\begin{aligned} (\mathbf{K}' - \mathbf{K}) &= \mathbf{K}\mathbf{m}_R\mathbf{n}_R^{\mathrm{T}}\mathbf{n}_L\mathbf{m}_L^{\mathrm{T}}\mathbf{K} + \mathbf{n}_L(\mathbf{m}_L^{\mathrm{T}}\mathbf{K}\mathbf{m}_R)\mathbf{n}_R^{\mathrm{T}}, \\ (\mathbf{D}' - \mathbf{D}) &= \mathbf{D}\mathbf{m}_R\mathbf{n}_R^{\mathrm{T}} + \mathbf{n}_L\mathbf{m}_L^{\mathrm{T}}\mathbf{D} + \mathbf{n}_L(\mathbf{m}_L^{\mathrm{T}}\mathbf{D}\mathbf{m}_R)\mathbf{n}_R^{\mathrm{T}}, \\ (\mathbf{M}' - \mathbf{M}) &= \mathbf{M}\mathbf{m}_R\mathbf{n}_R^{\mathrm{T}} + \mathbf{n}_L\mathbf{m}_L^{\mathrm{T}}\mathbf{M} + \mathbf{n}_L(\mathbf{m}_L^{\mathrm{T}}\mathbf{M}\mathbf{m}_R)\mathbf{n}_R^{\mathrm{T}}. \end{aligned} \tag{20}$$

Each of the above modifications can be shown to have rank 2.

4.2. CLASS-2 ELEMENTARY STRUCTURE-PRESERVING TRANSFORMATIONS

Equations (15) are used to eliminate $\{\mathbf{c}_L, \mathbf{e}_L, \mathbf{g}_L, \mathbf{c}_R, \mathbf{e}_R, \mathbf{g}_R\}$ from equations (11) and (12) so that the general Class-2 transformation is described by

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix} := \begin{bmatrix} (\mathbf{I} + \mathbf{a}_L\mathbf{b}_L^{\mathrm{T}}) & (\mathbf{a}_L\mathbf{d}_L^{\mathrm{T}}) \\ (\mathbf{a}_L\mathbf{f}_L^{\mathrm{T}}) & (\mathbf{I} + \mathbf{a}_L\mathbf{h}_L^{\mathrm{T}}) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} := \begin{bmatrix} (\mathbf{I} + \mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) & (\mathbf{a}_R\mathbf{d}_R^{\mathrm{T}}) \\ (\mathbf{a}_R\mathbf{f}_R^{\mathrm{T}}) & (\mathbf{I} + \mathbf{a}_R\mathbf{h}_R^{\mathrm{T}}) \end{bmatrix}. \tag{21}$$

The vectors are subject to the constraints of equation (16). The modifications occurring in the system matrices as a result of one of these Class-2 elementary transformations are found to be

$$\begin{aligned} (\mathbf{K} - \mathbf{K'}) &= x_K\mathbf{b}_L\mathbf{b}_R^{\mathrm{T}} + x_D(\mathbf{b}_L\mathbf{f}_R^{\mathrm{T}} + \mathbf{f}_L\mathbf{b}_R^{\mathrm{T}}) + x_M\mathbf{f}_L\mathbf{f}_R^{\mathrm{T}}, \\ (\mathbf{D} - \mathbf{D'}) &= x_K(\mathbf{b}_L\mathbf{d}_R^{\mathrm{T}} + \mathbf{d}_L\mathbf{b}_R^{\mathrm{T}}) \\ &\quad + x_D(\mathbf{b}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{b}_R^{\mathrm{T}} + \mathbf{f}_L\mathbf{d}_R^{\mathrm{T}} + \mathbf{d}_L\mathbf{f}_R^{\mathrm{T}}) + x_M(\mathbf{f}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{f}_R^{\mathrm{T}}), \\ (\mathbf{M} - \mathbf{M'}) &= x_K\mathbf{d}_L\mathbf{d}_R^{\mathrm{T}} + x_D(\mathbf{d}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{d}_R^{\mathrm{T}}) + x_M\mathbf{h}_L\mathbf{h}_R^{\mathrm{T}}. \end{aligned} \tag{22}$$

A derivation of these identities is provided in Appendix B. The modifications in equation (22) are deliberately presented in the opposite sense from those of equation (20) to avoid the need for additional brackets. The modifications to the system stiffness and mass matrices under a Class-2 transformation are each rank 2. The modification to the system damping matrix is rank 4.

5. THE INVERSES OF THE ELEMENTARY TRANSFORMATIONS

For every structure-preserving transformation which transforms the system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ into the new system $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$, there must be an inverse structure-preserving transformation which transforms back from system $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ to $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$. The inverses of the elementary structure-preserving transformations are of interest for two separate reasons:

- firstly, considerable insight is gained into the condition of the transformation from the expression of its inverse and it becomes evident how to construct well-conditioned transformations;
- secondly, it is interesting to discover that the inverse of an elementary structure-preserving transformation is an elementary structure-preserving transformation of the same form.

In all previous sections, it has been recognized that the transformation applied to the left of the system may be different from that applied to the right. The distinction between left and right transformations is not relevant here and subscripts R and L are dropped.

Class-1 elementary structure-preserving transformations involve matrices of the structure $(\mathbf{I}+\mathbf{mn}^{\mathrm{T}})$.

The inverse of any one such transformation is clear from

$$(\mathbf{I}+\mathbf{mn}^{\mathrm{T}})^{-1} = (\mathbf{I}-(\mathbf{mn}^{\mathrm{T}})/(1+\mathbf{m}^{\mathrm{T}}\mathbf{n})). \tag{23}$$

Evidently, a given Class-1 transformation matrix will be well conditioned provided that $(1+\mathbf{m}^{\mathrm{T}}\mathbf{n})$ is not close to zero.

The inverse of the general Class-2 elementary structure-preserving transformation matrix is more challenging. Postulating that the inverse has the same form as the transformation itself, it is evident that there must be some vectors $\{\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\}$ satisfying

$$\begin{bmatrix} (\mathbf{I}+\mathbf{rs}^{\mathrm{T}}) & (\mathbf{rt}^{\mathrm{T}}) \\ (\mathbf{ru}^{\mathrm{T}}) & (\mathbf{I}+\mathbf{rv}^{\mathrm{T}}) \end{bmatrix} \begin{bmatrix} (\mathbf{I}+\mathbf{ab}^{\mathrm{T}}) & (\mathbf{ad}^{\mathrm{T}}) \\ (\mathbf{af}^{\mathrm{T}}) & (\mathbf{I}+\mathbf{ah}^{\mathrm{T}}) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{24}$$

Appendix C shows how the vectors $\{\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\}$ are computed. An immediate conclusion from the expansion of equation (24) is that

$$\mathbf{r} = \mathbf{a}, \tag{25}$$

without loss of generality. It is also immediately evident that vectors $\{\mathbf{s}, \mathbf{u}\}$ are each linear combinations of $\{\mathbf{b}, \mathbf{f}\}$ and that vectors $\{\mathbf{t}, \mathbf{v}\}$ are each linear combinations of $\{\mathbf{d}, \mathbf{h}\}$. The detail behind the determination of coefficients is consigned to Appendix B so as not to obscure the simplicity of this important point:

The only criterion which can lead to a poorly conditioned inverse of a Class-2 elementary structure preserving transformation matrix is

$$(1+\mathbf{b}^{\mathrm{T}}\mathbf{a})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) - (\mathbf{d}^{\mathrm{T}}\mathbf{a})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) = 0. \tag{26}$$

In devising Class-2 transformations for specific purposes, numerical stability demands that they should be far removed from the possibility that this might occur.

6. ILLUSTRATION

The matter of how to determine a series of elementary structure-preserving transformations such that the system matrices at the end of this series of transformations have a banded structure or a bordered–banded structure reduces to the following single question:

Given a system characterized by matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, how can structure-preserving co-ordinate transformation matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ be computed directly such that

- the transformation represented by $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ is a low-rank modification of the identity transformation and it is well conditioned,
- the transformed matrices $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ each have zeros in all positions in the first row and first column except in positions (1,1), (1,2) and (2,1),
- the transformation matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ each have zeros in all positions in row 1 except position (1,1) which should equal unity,
- the transformation represented by $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ is optimally conditioned?

Such transformations will be a generalization of transformations which have been reported by the present authors [6] for the simultaneous tridiagonalization of two symmetric matrices.

The answer to this question is not yet fully resolved but the indications are that transformations comprising a Class-2 component followed by a Class-1 component (or *vice versa*) will serve for this purpose. The number of independent parameters in a Class-2 transformation $(4N-2)$ alone is not sufficient to satisfy this requirement. This same answer will provide the tool with which bulge chasing can be conducted and therefore it will simultaneously provide a set of routes to the solution of the quadratic eigenvalue problem which do not involve the conventional solution for complex modes. An illustration is given as an indication of what is to come.

The following illustration has been computed beginning from a system characterized by tridiagonal matrices and using arbitrary Class-2 elementary transformations to "un-tridiagonalize" it in steps. No Class-1 elementary transformations are used here. Since the inverse of any elementary structure-preserving transformation ($\{\mathbf{K}, \mathbf{D}, \mathbf{M}\} \rightarrow \{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$) is the elementary structure-preserving transformation ($\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\} \rightarrow \{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$) having the same form, constructing an illustration of system tridiagonalisation is not difficult. For the purposes of illustration, the example is presented beginning with the fully populated symmetric system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$, and then showing increasingly structured systems $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$, $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$ and $\{\mathbf{K}_4, \mathbf{D}_4, \mathbf{M}_4\}$ (all symmetric) with the latter system being tridiagonal.

Because of the symmetry, no distinction is needed between the left and right transformations. The vectors describing each of the three different Class-2 elementary transformations applied are also reported. Vectors $\{\mathbf{a}_{12}, \mathbf{b}_{12}, \mathbf{d}_{12}, \mathbf{f}_{12}, \mathbf{h}_{12}\}$ describe the transformation between $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ and $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$, vectors $\{\mathbf{a}_{23}, \mathbf{b}_{23}, \mathbf{d}_{23}, \mathbf{f}_{23}, \mathbf{h}_{23}\}$ describe the transformation between $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ and $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$ and vectors $\{\mathbf{a}_{34}, \mathbf{b}_{34}, \mathbf{d}_{34}, \mathbf{f}_{34}, \mathbf{h}_{34}\}$ describe the transformation between $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$ and $\{\mathbf{K}_4, \mathbf{D}_4, \mathbf{M}_4\}$.

All matrices and vectors are reported to a high degree of precision so that the process can be reproduced accurately:

$$\mathbf{M}_1 = \begin{bmatrix} 2{\cdot}514552538440\text{E}+0 & 6{\cdot}796559447073\text{E}-1 & 5{\cdot}609999855327\text{E}-1 & 1{\cdot}059687009842\text{E}-1 & -4{\cdot}299655673920\text{E}-1 \\ 6{\cdot}796559447073\text{E}-1 & 1{\cdot}189679145862\text{E}+0 & 8{\cdot}806062340494\text{E}-1 & 3{\cdot}448610219661\text{E}-1 & -6{\cdot}358515616947\text{E}-1 \\ 5{\cdot}609999855327\text{E}-1 & 8{\cdot}806062340494\text{E}-1 & 5{\cdot}994678345702\text{E}-1 & 3{\cdot}206487491067\text{E}-1 & -4{\cdot}113543251574\text{E}-1 \\ 1{\cdot}059687009842\text{E}-1 & 3{\cdot}448610219661\text{E}-1 & 3{\cdot}206487491067\text{E}-1 & 1{\cdot}660977161872\text{E}-2 & -2{\cdot}529474024828\text{E}-1 \\ -4{\cdot}299655673920\text{E}-1 & -6{\cdot}358515616947\text{E}-1 & -4{\cdot}113543251574\text{E}-1 & -2{\cdot}529474024828\text{E}-1 & 2{\cdot}212367731216\text{E}-1 \end{bmatrix}, \tag{27}$$

$$\mathbf{C}_1 = \begin{bmatrix} 1{\cdot}019268190703\text{E}+1 & -1{\cdot}380449665411\text{E}+0 & -1{\cdot}139160405307\text{E}+0 & -2{\cdot}144380577136\text{E}-1 & 8{\cdot}507628714758\text{E}-1 \\ -1{\cdot}380449665411\text{E}+0 & 4{\cdot}930314048456\text{E}+0 & 3{\cdot}556761340894\text{E}+0 & 1{\cdot}467973391250\text{E}+0 & -2{\cdot}265758694548\text{E}+0 \\ -1{\cdot}139160405307\text{E}+0 & 3{\cdot}556761340894\text{E}+0 & 2{\cdot}511616346633\text{E}+0 & 1{\cdot}131979877414\text{E}+0 & -1{\cdot}580855438285\text{E}+0 \\ -2{\cdot}144380577136\text{E}-1 & 1{\cdot}467973391250\text{E}+0 & 1{\cdot}131979877414\text{E}+0 & 3{\cdot}367789544473\text{E}-1 & -7{\cdot}323644438404\text{E}-1 \\ 8{\cdot}507628714758\text{E}-1 & -2{\cdot}265758694548\text{E}+0 & -1{\cdot}580855438285\text{E}+0 & -7{\cdot}323644438404\text{E}-1 & 8{\cdot}635728513120\text{E}-1 \end{bmatrix}, \tag{28}$$

$$\mathbf{K}_1 = \begin{bmatrix} 3{\cdot}434210664432\text{E}+1 & -3{\cdot}100846685418\text{E}+0 & -1{\cdot}769917339996\text{E}+0 & -1{\cdot}539222788457\text{E}+0 & 4{\cdot}910446683326\text{E}-1 \\ -3{\cdot}100846685418\text{E}+0 & 4{\cdot}361433462891\text{E}+0 & 3{\cdot}081163329822\text{E}+0 & 1{\cdot}465446904886\text{E}+0 & -1{\cdot}978438484593\text{E}+0 \\ -1{\cdot}769917339996\text{E}+0 & 3{\cdot}081163329822\text{E}+0 & 1{\cdot}906016577297\text{E}+0 & 1{\cdot}372984280593\text{E}+0 & -1{\cdot}027663084239\text{E}+0 \\ -1{\cdot}539222788457\text{E}+0 & 1{\cdot}465446904886\text{E}+0 & 1{\cdot}372984280593\text{E}+0 & 5{\cdot}800797402211\text{E}-2 & -1{\cdot}067203233125\text{E}+0 \\ 4{\cdot}910446683326\text{E}-1 & -1{\cdot}978438484593\text{E}+0 & -1{\cdot}027663084239\text{E}+0 & -1{\cdot}067203233125\text{E}+0 & 1{\cdot}806904351274\text{E}-2 \end{bmatrix}, \tag{29}$$

$$[\mathbf{a}_{12}\mathbf{b}_{12}\mathbf{d}_{12}\mathbf{f}_{12}\mathbf{h}_{12}] = \begin{bmatrix} 0{\cdot}000000000000\text{E}+0 & -8{\cdot}818008634402\text{E}+1 & 1{\cdot}841672054610\text{E}+1 & 1{\cdot}289170438227\text{E}+2 & -6{\cdot}976336579791\text{E}+1 \\ -2{\cdot}788982905733\text{E}-1 & 6{\cdot}425652524803\text{E}+1 & 3{\cdot}765267040090\text{E}+1 & -2{\cdot}529421720610\text{E}+2 & -3{\cdot}036180015698\text{E}+1 \\ 3{\cdot}175431571666\text{E}-1 & 4{\cdot}381078920910\text{E}+0 & -7{\cdot}345899496798\text{E}+1 & 2{\cdot}451348343741\text{E}+2 & 4{\cdot}854826590775\text{E}+1 \\ 2{\cdot}303887956148\text{E}-1 & -2{\cdot}784205905415\text{E}+1 & -2{\cdot}837213130100\text{E}+1 & 9{\cdot}023202474142\text{E}+1 & 2{\cdot}595080091807\text{E}-1 \\ 4{\cdot}416117392699\text{E}-2 & -2{\cdot}358337461023\text{E}+1 & 4{\cdot}470402951462\text{E}+1 & -1{\cdot}548973144262\text{E}+2 & -4.433736649220\text{E}+1 \end{bmatrix}, \tag{30}$$

$$\mathbf{M}_2 = \begin{bmatrix} 3{\cdot}000000000000E+0 & 1{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 1{\cdot}000000000000E+0 & 1{\cdot}755158164336E+0 & -2{\cdot}141074791914E-1 & -6{\cdot}321671596836E-2 & 4{\cdot}213228563312E-2 \\ 0{\cdot}000000000000E+0 & -2{\cdot}141074791914E-1 & 2{\cdot}730241908707E+0 & 1{\cdot}135014993984E+0 & -1{\cdot}714936602206E+0 \\ 0{\cdot}000000000000E+0 & -6{\cdot}321671596836E-2 & 1{\cdot}135014993984E+0 & 3{\cdot}626723938057E-1 & -7{\cdot}233129697536E-1 \\ 0{\cdot}000000000000E+0 & 4{\cdot}213228563312E-2 & -1{\cdot}714936602206E+0 & -7{\cdot}233129697536E-1 & 1{\cdot}041035984013E+0 \end{bmatrix}, \tag{31}$$

$$\mathbf{D}_2 = \begin{bmatrix} 8{\cdot}000000000000E+0 & -1{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ -1{\cdot}000000000000E+0 & 7{\cdot}180891786074E+0 & 1{\cdot}462627371105E+0 & 6{\cdot}423535886181E-1 & -9{\cdot}641185983009E-1 \\ 0{\cdot}000000000000E+0 & 1{\cdot}462627371105E+0 & 2{\cdot}488287707049E+0 & 1{\cdot}501519513906E+0 & -1{\cdot}342598058385E+0 \\ 0{\cdot}000000000000E+0 & 6{\cdot}423535886181E-1 & 1{\cdot}501519513906E+0 & 7{\cdot}091356957399E-1 & -8{\cdot}029592159905E-1 \\ 0{\cdot}000000000000E+0 & -9{\cdot}641185983009E-1 & -1{\cdot}342598058385E+0 & -8{\cdot}029592159905E-1 & 6{\cdot}372396410684E-1 \end{bmatrix}, \tag{32}$$

$$\mathbf{K}_2 = \begin{bmatrix} 3{\cdot}800000000000E+1 & -7{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ -7{\cdot}000000000000E+0 & 1{\cdot}067859107614E+1 & -2{\cdot}223935371196E+0 & -8{\cdot}483810467368E-1 & 1{\cdot}118621601203E+0 \\ 0{\cdot}000000000000E+0 & -2{\cdot}223935371196E+0 & 8{\cdot}170406649053E+0 & 3{\cdot}182578393286E+0 & -5{\cdot}336751293275E+0 \\ 0{\cdot}000000000000E+0 & -8{\cdot}483810467368E-1 & 3{\cdot}182578393286E+0 & 9{\cdot}153661297797E-1 & -2{\cdot}075499718163E+0 \\ 0{\cdot}000000000000E+0 & 1{\cdot}118621601203E+0 & -5{\cdot}336751293275E+0 & -2{\cdot}075499718163E+0 & 3{\cdot}149271337742E+0 \end{bmatrix}, \tag{33}$$

$$[\mathbf{a}_{23}\mathbf{b}_{23}\mathbf{d}_{23}\mathbf{f}_{23}\mathbf{h}_{23}] = \begin{bmatrix} 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & -1{\cdot}386533382617E+0 & 5{\cdot}4400928207E-015 & 3{\cdot}267306243365E+0 & -3{\cdot}564737544860E+0 \\ 5{\cdot}103780278271E-1 & 1{\cdot}409996801975E+0 & 3{\cdot}329462938489E+0 & -2{\cdot}102754287758E+1 & -7{\cdot}800908465912E+0 \\ -1{\cdot}381631989881E-1 & -2{\cdot}630586887750E+0 & -9{\cdot}555380110430E-1 & 1{\cdot}560243930860E+1 & 4{\cdot}237862806769E-1 \\ 7{\cdot}561217117064E-1 & 3{\cdot}064909179669E+0 & 1{\cdot}715198076126E+0 & -2{\cdot}330881101331E+1 & -1{\cdot}799395645483E+0 \end{bmatrix}, \tag{34}$$

$$\mathbf{M}_3 = \begin{bmatrix} 3{\cdot}000000000000\mathrm{E}+0 & 1{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ 1{\cdot}000000000000\mathrm{E}+0 & 2{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 4{\cdot}578117891291\mathrm{E}+0 & 7{\cdot}137001110964\mathrm{E}-1 & -8{\cdot}962863435737\mathrm{E}-1 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 7{\cdot}137001110964\mathrm{E}-1 & 5{\cdot}157751217053\mathrm{E}-1 & -9{\cdot}797134072716\mathrm{E}-1 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & -8{\cdot}962863435737\mathrm{E}-1 & -9{\cdot}797134072716\mathrm{E}-1 & 1{\cdot}489000202262\mathrm{E}+0 \end{bmatrix}, \quad (35)$$

$$\mathbf{D}_3 = \begin{bmatrix} 8{\cdot}000000000000\mathrm{E}+0 & -1{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ -1{\cdot}000000000000\mathrm{E}+0 & 7{\cdot}000000000000\mathrm{E}+0 & 2{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 2{\cdot}000000000000\mathrm{E}+0 & 6{\cdot}182495923235\mathrm{E}+0 & -3{\cdot}895143143096\mathrm{E}-1 & 1{\cdot}464332065665\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & -3{\cdot}895143143096\mathrm{E}-1 & 1{\cdot}040111572147\mathrm{E}+0 & -1{\cdot}440038145687\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 1{\cdot}464332065665\mathrm{E}+0 & -1{\cdot}440038145687\mathrm{E}+0 & 1{\cdot}763209234891\mathrm{E}+0 \end{bmatrix}, \quad (36)$$

$$\mathbf{K}_3 = \begin{bmatrix} 3{\cdot}800000000000\mathrm{E}+1 & -7{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ -7{\cdot}000000000000\mathrm{E}+0 & 1{\cdot}100000000000\mathrm{E}+1 & -3{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & -3{\cdot}000000000000\mathrm{E}+0 & 1{\cdot}545618323018\mathrm{E}+1 & -1{\cdot}740311056911\mathrm{E}+0 & 2{\cdot}281434304519\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & -1{\cdot}740311056911\mathrm{E}+0 & 4{\cdot}681404919161\mathrm{E}+0 & -7{\cdot}639833302348\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 2{\cdot}281434304519\mathrm{E}+0 & -7{\cdot}639833302348\mathrm{E}+0 & 1{\cdot}151018620998\mathrm{E}+1 \end{bmatrix}, \quad (37)$$

$$[\mathbf{a}_{34}\mathbf{b}_{34}\mathbf{d}_{34}\mathbf{f}_{34}\mathbf{h}_{34}] = \begin{bmatrix} 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 & 0{\cdot}000000000000\mathrm{E}+0 \\ 0{\cdot}000000000000\mathrm{E}+0 & 2{\cdot}057195824880\mathrm{E}+0 & -2{\cdot}361632597869\mathrm{E}+0 & -4{\cdot}723265195739\mathrm{E}+0 & -1{\cdot}485253071924\mathrm{E}+0 \\ -3{\cdot}888460332497\mathrm{E}-1 & -8{\cdot}477660971198\mathrm{E}-1 & -1{\cdot}061986006121\mathrm{E}+1 & 2{\cdot}832461791603\mathrm{E}+1 & 6{\cdot}625746043331\mathrm{E}+0 \\ -2{\cdot}584088996830\mathrm{E}-1 & -3{\cdot}227677448428\mathrm{E}+0 & -6{\cdot}681310188359\mathrm{E}+0 & 2{\cdot}591806554294\mathrm{E}+1 & 2{\cdot}272816440996\mathrm{E}+0 \end{bmatrix}, \quad (38)$$

$$\mathbf{M}_4 = \begin{bmatrix} 3{\cdot}000000000000E+0 & 1{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 1{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 5{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 & 7{\cdot}000000000000E+0 & 3{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 3{\cdot}000000000000E+0 & 4{\cdot}000000000000E+0 \end{bmatrix}, \qquad (39)$$

$$\mathbf{D}_4 = \begin{bmatrix} 8{\cdot}000000000000E+0 & -1{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ -1{\cdot}000000000000E+0 & 7{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 & 6{\cdot}000000000000E+0 & -3{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & -3{\cdot}000000000000E+0 & 5{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 2{\cdot}000000000000E+0 & 4{\cdot}000000000000E+0 \end{bmatrix}, \quad (40)$$

$$\mathbf{K}_4 = \begin{bmatrix} 3{\cdot}800000000000E+1 & -7{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ -7{\cdot}000000000000E+0 & 1{\cdot}100000000000E+1 & -3{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & -3{\cdot}000000000000E+0 & 1{\cdot}600000000000E+1 & -4{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & -4{\cdot}000000000000E+0 & 2{\cdot}000000000000E+1 & 6{\cdot}000000000000E+0 \\ 0{\cdot}000000000000E+0 & 0{\cdot}000000000000E+0 & -0{\cdot}000000000000E+0 & 6{\cdot}000000000000E+0 & 2{\cdot}400000000000E+1 \end{bmatrix}. \quad (41)$$

7. CONCLUSIONS

This paper defines the concept of an elementary structure-preserving transformation for a general second order system and justifies the study of these by highlighting the usefulness of different structures of matrix in different areas of vibration analysis.

It shows that there are two distinct classes of elementary structure-preserving transformations for second order systems. One of these classes (Class-1) contains all of the "conventional" co-ordinate transformation matrices that are unit-rank modifications of the identity transformation. Transformations within the other class involve combinations of displacements and velocities from the original system co-ordinates in the displacement co-ordinates of the transformed system. For a second order system having N degrees of freedom, the dimension of the space of all Class-1 elementary transformations is $(4N–2)$. The dimension of the space of all Class-2 elementary structure-preserving transformations is also $(4N–2)$.

The paper shows how the inverses of elementary transformations from the two different classes are computed and from this, it extracts the unique condition which must apply if one of the elementary transformations is to be singular. Obviously, this is a condition to be avoided widely. Finally, an illustration is given of the simultaneous tridiagonalization of three system matrices of a symmetric system.

The implications of this paper range over numerical processes in vibration analysis, theoretical reasoning and practical development and use of methods.

In the category of contributions to numerical processes, the ability to transform from a general (self-adjoint) system to a tridiagonal form and subsequently to a diagonal form may completely supplant the existing numerical methods for computing eigenfrequencies and modes. Such a transformation is expected to follow a close parallel to a process already developed for simultaneously tridiagonalizing the two matrices of an undamped system [6]. Tridiagonal system form may be especially useful for dynamic substructuring applications. The possibility of performing structure-preserving model-reducing transformations constructed from elementary structure-preserving transformations in the course of a finite-element calculation (model assembly) also appears very strong.

In the category of theoretical reasoning, it is immediately evident that if it can be proven that well-conditioned elementary transformations exist which can perform any one step of the system tridiagonalization process, then Falk's theorem [7] that every undamped system is equivalent to a chain system may be generalised to damped systems.

There are numerous possibilities in the category of practical methods development and use and these include applications in active vibration control, system identification and updating and efficient compression/expansion of "complex" modal data.

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APPENDIX A: DERIVATION OF THE STRUCTURE-PRESERVING CONSTRAINTS

Equations (11) and (12) in the main text can be used to substitute for $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ in equations (5), (8) and (9) to produce constraint equations acting on $\{\mathbf{a}_L, \mathbf{b}_L, \mathbf{c}_L, \mathbf{d}_L, \mathbf{e}_L, \mathbf{f}_L, \mathbf{g}_L, \mathbf{h}_L\}$ and $\{\mathbf{a}_R, \mathbf{b}_R, \mathbf{c}_R, \mathbf{d}_R, \mathbf{e}_R, \mathbf{f}_R, \mathbf{g}_R, \mathbf{h}_R\}$. Equation (5) of the main text produces

$$\begin{aligned}
&(\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{I}+\mathbf{g}_R\mathbf{h}_R^{\mathrm{T}}) + \mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}(\mathbf{K}\mathbf{c}_R\mathbf{d}_R^{\mathrm{T}} + \mathbf{D}(\mathbf{I}+\mathbf{g}_R\mathbf{h}_R^{\mathrm{T}}))\\
&\quad = (\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})^{\mathrm{T}}\mathbf{K}(\mathbf{I}+\mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) - \mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}\mathbf{M}\mathbf{e}_R\mathbf{f}_R,\\
&(\mathbf{I}+\mathbf{h}_L\mathbf{g}_L^{\mathrm{T}})\mathbf{K}(\mathbf{I}+\mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) + (\mathbf{d}_L\mathbf{c}_L^{\mathrm{T}}\mathbf{K} + (\mathbf{I}+\mathbf{g}_L\mathbf{h}_L^{\mathrm{T}})\mathbf{D})\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}}\\
&\quad = (\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})^{\mathrm{T}}\mathbf{K}(\mathbf{I}+\mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) - \mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}\mathbf{M}\mathbf{e}_R\mathbf{f}_R.
\end{aligned} \tag{A.1}$$

Equation (8) of the main text reveals

$$\mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}\mathbf{D}\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}} + (\mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}\mathbf{K}(\mathbf{I}+\mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) + (\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}}) = 0 \tag{A.2}$$

and equation (9) of the main text emerges as

$$\begin{aligned}
&(\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{c}_R\mathbf{d}_R^{\mathrm{T}} - \mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}\mathbf{M}(\mathbf{I}+\mathbf{g}_R\mathbf{h}_R^{\mathrm{T}}) = 0,\\
&\mathbf{d}_L\mathbf{c}_L^{\mathrm{T}}\mathbf{K}(\mathbf{I}+\mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) - (\mathbf{I}+\mathbf{h}_L\mathbf{g}_L^{\mathrm{T}})\mathbf{M}\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}} = 0.
\end{aligned} \tag{A.3}$$

The concern of this Appendix is to distill from the quadratic conditions of equations (A.1)–(A.3), linear conditions on the vectors. This process begins by rewriting equation (A.2) as follows:

$$\begin{aligned}
&(\mathbf{f}_L + \alpha^{-1}((\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{e}_R))\alpha(\mathbf{f}_R + \alpha^{-1}((\mathbf{I}+\mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{K}^T\mathbf{e}_L))^{\mathrm{T}}\\
&\quad = ((\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{e}_R)\alpha^{-1}((\mathbf{I}+\mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{K}\mathbf{e}_L)^{\mathrm{T}},\\
&\text{where } \alpha = \mathbf{e}_L^{\mathrm{T}}\mathbf{D}\mathbf{e}_R.
\end{aligned} \tag{A.4}$$

There are two distinct ways in which equation (A.4) may be satisfied. Each of these leads to a distinct set of solutions. The trivial solutions to equation (A.4) arise by setting:

$$\mathbf{f}_L = 0 = \mathbf{f}_R \tag{A.5}$$

The elementary structure-preserving transformations satisfying equation (A.5) will be referred to as Class-1 elementary structure-preserving transformations. The term Class-2 elementary structure-preserving transformations will be used to describe those elementary structure-preserving transformations obeying

$$\mathbf{f}_L = -2\alpha^{-1}((\mathbf{I} + \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{e}_R), \quad \mathbf{f}_R = -2\alpha^{-1}((\mathbf{I} + \mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{K}^T\mathbf{e}_L). \tag{A.6}$$

Now gather terms in equation (A.1) with the following result:

$$\begin{aligned}(\mathbf{I} + \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{g}_R\mathbf{h}_R^{\mathrm{T}} - \mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) + \mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}(\mathbf{K}\mathbf{c}_R\mathbf{d}_R^{\mathrm{T}} + \mathbf{D}(\mathbf{I} + \mathbf{g}_R\mathbf{h}_R^{\mathrm{T}}) + \mathbf{M}\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}}) = 0,\\ (\mathbf{h}_L\mathbf{g}_L^{\mathrm{T}} - \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{I} + \mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) + (\mathbf{d}_L\mathbf{c}_L^{\mathrm{T}}\mathbf{K} + (\mathbf{I} + \mathbf{h}_L\mathbf{g}_L^{\mathrm{T}})\mathbf{D} + \mathbf{f}_L\mathbf{e}_L^{\mathrm{T}}\mathbf{M})\mathbf{e}_R\mathbf{f}_R^{\mathrm{T}} = 0.\end{aligned} \tag{A.7}$$

For Class-1 transformations, equation (A.5) applies and if it is assumed that matrices $\{\mathbf{K}, (\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}}), (\mathbf{I}+\mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\}$ are all non-singular then it is necessary that

$$\mathbf{g}_R\mathbf{h}_R^{\mathrm{T}} = \mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}, \qquad \mathbf{g}_L\mathbf{h}_L^{\mathrm{T}} = \mathbf{a}_L\mathbf{b}_L^{\mathrm{T}}. \tag{A.8}$$

Retaining the assumption that $\{\mathbf{K}, (\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}}), (\mathbf{I}+\mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\}$ are all non-singular, equation (A.3) then demands

$$\mathbf{c}_R\mathbf{d}_R^{\mathrm{T}} = \mathbf{0} = \mathbf{c}_L\mathbf{d}_L^{\mathrm{T}}. \tag{A.9}$$

Additional solutions may be possible in special cases where $(\mathbf{I}+\mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})$ and/or $(\mathbf{I}+\mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})$ are singular but this would mean that one or both of the transformation matrices were singular and such cases are of no practical importance or use. Similarly, additional solutions might be possible when $\mathbf{K}$ is singular but there appears to be little practical motivation for investigating these.

Equations (A.5), (A.8) and (A.9) can be compacted, without loss of generality, into the following linear constraints applying to the general Class-1 elementary structure-preserving transformation:

$$\begin{aligned}\mathbf{g}_L = \mathbf{a}_L, \quad \mathbf{h}_L = \mathbf{b}_L, \quad \mathbf{c}_L = \mathbf{d}_L = \mathbf{e}_L = \mathbf{f}_L = \mathbf{0},\\ \mathbf{g}_R = \mathbf{a}_R, \quad \mathbf{h}_R = \mathbf{b}_R, \quad \mathbf{c}_R = \mathbf{d}_R = \mathbf{e}_R = \mathbf{f}_R = \mathbf{0}.\end{aligned} \tag{A.10}$$

Now consider that equation (A.6) holds in place of equation (A.5) as a means of ensuring that equation (A.2) is satisfied. Equations (A.3) are satisfied if there are some real scalars, β_1 and β_2, such that

$$\begin{aligned}\mathbf{f}_L = \beta_1(\mathbf{I} + \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{c}_R, \quad \mathbf{d}_R = \beta_1(\mathbf{I} + \mathbf{h}_R\mathbf{g}_R^{\mathrm{T}})\mathbf{M}^{\mathrm{T}}\mathbf{e}_L,\\ \mathbf{f}_R = \beta_2(\mathbf{I} + \mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{K}^{\mathrm{T}}\mathbf{c}_L, \quad \mathbf{d}_L = \beta_2(\mathbf{I} + \mathbf{h}_L\mathbf{g}_L^{\mathrm{T}})\mathbf{M}\mathbf{e}_R.\end{aligned} \tag{A.11}$$

Because $\mathbf{c}_R$ and $\mathbf{d}_R$ appear together in the product $\mathbf{c}_R\mathbf{d}_R^{\mathrm{T}}$ and $\mathbf{c}_L$ and $\mathbf{d}_L$ appear together in the product $\mathbf{c}_L\mathbf{d}_L^{\mathrm{T}}$, there is no loss of generality in setting

$$\beta_1 = -2\alpha^{-1} = \beta_2 \tag{A.12}$$

in which case a direct comparison of equations (A.11) and (A.6) produces the linear constraints

$$\mathbf{c}_L = \mathbf{e}_L, \qquad \mathbf{c}_R = \mathbf{e}_R. \tag{A.13}$$

Now examine equations (A.7) again. The second term in each of these equations is unit rank. It follows that the first term must also be unit rank in each case. There are two possible ways to ensure that the first term is single rank: either

$$\mathbf{h}_R = \mathbf{b}_R\gamma_1, \quad \mathbf{h}_L = \mathbf{b}_L\gamma_2 \tag{A.14}$$

for some arbitrary real constants, γ_1 and γ_2, or

$$\mathbf{g}_R = \mathbf{a}_R\gamma_1, \qquad \mathbf{g}_L = \mathbf{a}_L\gamma_2. \tag{A.15}$$

The Class-1 elementary structure-preserving transformations obey equation (A.14). For Class-2 elementary structure-preserving transformations, equations (A.15) are applied and the scalars γ_1 and γ_2 are set to unity

$$\gamma_1 = 1 = \gamma_2 \tag{A.16}$$

without loss of generality. Then equations (A.7) can be satisfied by

$$\begin{aligned}
&\mathbf{f}_L = \delta_1(\mathbf{I} + \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{a}_R,\\
&(\mathbf{h}_R - \mathbf{b}_R) = -\delta_1[\mathbf{d}_R\mathbf{c}_R^{\mathrm{T}}\mathbf{K}^{\mathrm{T}} + (\mathbf{I} + \mathbf{h}_R\mathbf{g}_R^{\mathrm{T}})\mathbf{D}^{\mathrm{T}} + \mathbf{f}_R\mathbf{e}_R^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}]\mathbf{e}_L,\\
&\mathbf{f}_R = \delta_2(\mathbf{I} + \mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{K}^{\mathrm{T}}\mathbf{a}_L,\\
&(\mathbf{h}_L - \mathbf{b}_L) = -\delta_2[\mathbf{d}_L\mathbf{c}_L^{\mathrm{T}}\mathbf{K} + (\mathbf{I} + \mathbf{h}_L\mathbf{g}_L^{\mathrm{T}})\mathbf{D} + \mathbf{f}_L\mathbf{e}_L^{T}\mathbf{M}]\mathbf{e}_R.
\end{aligned} \tag{A.17}$$

where δ_1 and δ_2 are arbitrary real constants. Once again, without loss of generality, set

$$\delta_1 = -2\alpha^{-1} = \delta_2 \tag{A.18}$$

Then comparing equations (A.17) and (A.11) with equation (A.6) and recalling equation (A.15) yields

$$\mathbf{g}_L = \mathbf{e}_L = \mathbf{c}_L = \mathbf{a}_L, \qquad \mathbf{g}_R = \mathbf{e}_R = \mathbf{c}_R = \mathbf{a}_R. \tag{A.19}$$

Use equation (A.19) to eliminate $\{\mathbf{c}_L, \mathbf{e}_L, \mathbf{g}_L, \mathbf{c}_R, \mathbf{e}_R, \mathbf{g}_R\}$ in all previous equations and define

$$x_K := (\mathbf{a}_L^{\mathrm{T}}\mathbf{K}\mathbf{a}_R), \quad x_D := \tfrac{1}{2}(\mathbf{a}_L^{\mathrm{T}}\mathbf{D}\mathbf{a}_R), \quad x_M := (\mathbf{a}_L^{\mathrm{T}}\mathbf{M}\mathbf{a}_R). \tag{A.20}$$

Equations (A.11) then simplify to

$$\begin{aligned}
&\mathbf{f}_L x_D = -(\mathbf{I} + \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}\mathbf{a}_R, \quad \mathbf{d}_R x_D = -(\mathbf{I} + \mathbf{h}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{M}^{\mathrm{T}}\mathbf{a}_L,\\
&\mathbf{f}_R x_D = -(\mathbf{I} + \mathbf{b}_R\mathbf{a}_R^{\mathrm{T}})\mathbf{K}^{\mathrm{T}}a_L, \quad \mathbf{d}_L x_D = -(\mathbf{I} + \mathbf{h}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{M}\mathbf{a}_R.
\end{aligned} \tag{A.21}$$

Using equation (A.20), these can be rearranged to produce

$$\mathbf{K}\mathbf{a}_R + \mathbf{b}_L x_K + \mathbf{f}_L x_D = \mathbf{0}, \qquad \mathbf{K}^{\mathrm{T}}\mathbf{a}_L + \mathbf{b}_R x_K + \mathbf{f}_R x_D = \mathbf{0}. \tag{A.22}$$

and

$$\mathbf{M}\mathbf{a}_R + \mathbf{d}_L x_D + \mathbf{h}_L x_M = \mathbf{0}, \qquad \mathbf{M}^{\mathrm{T}}\mathbf{a}_L + \mathbf{d}_R x_D + \mathbf{h}_R x_M = \mathbf{0}. \tag{A.23}$$

Apply equation (A.21) to substitute for $\{\mathbf{d}_R, \mathbf{d}_L, \mathbf{f}_R, \mathbf{f}_L\}$ in the expressions for ($\mathbf{h}_L$–$\mathbf{b}_L$) and ($\mathbf{h}_R$–$\mathbf{b}_R$) in equation (A.17) with the result

$$\begin{aligned}
(\mathbf{b}_R + \mathbf{h}_R) &= -\frac{(\mathbf{M}^{\mathrm{T}}\mathbf{a}_L x_K - \mathbf{D}^{\mathrm{T}}\mathbf{a}_L x_D + \mathbf{K}^{\mathrm{T}}\mathbf{a}_L x_M)}{(x_K x_M - x_D^2)},\\
(\mathbf{b}_L + \mathbf{h}_L) &= -\frac{(\mathbf{M}\mathbf{a}_R x_K - \mathbf{D}\mathbf{a}_R x_D + \mathbf{K}\mathbf{a}_R x_M)}{(x_K x_M - x_D^2)}.
\end{aligned} \tag{A.24}$$

Multiply equations (A.22) by x_M, multiply equations (A.23) by x_K and add the two sets of equations together in order to obtain

$$\begin{aligned}
&(x_K x_M)(\mathbf{b}_L + \mathbf{h}_L) + x_M\mathbf{K}\mathbf{a}_R + (x_D x_M)\mathbf{f}_L + x_K\mathbf{M}\mathbf{a}_R + (x_K x_D)\mathbf{d}_L = \mathbf{0},\\
&(x_K x_M)(\mathbf{b}_R + \mathbf{h}_R) + x_M\mathbf{K}^{\mathrm{T}}\mathbf{a}_L + (x_D x_M)\mathbf{f}_R + x_K\mathbf{M}^{\mathrm{T}}\mathbf{a}_L + (x_K x_D)\mathbf{d}_R = \mathbf{0}.
\end{aligned} \tag{A.25}$$

Apply equation (A.24) to substitute for ($\mathbf{K}\mathbf{a}_R x_M + \mathbf{M}\mathbf{a}_R x_K$) and ($\mathbf{K}^{\mathrm{T}}\mathbf{a}_L x_M + \mathbf{M}^{\mathrm{T}}\mathbf{a}_L x_K$) in the above and it is found that

$$\begin{aligned}
&\mathbf{D}\mathbf{a}_R + \mathbf{d}_L x_K + (\mathbf{b}_L + \mathbf{h}_L)x_D + \mathbf{f}_L x_M = \mathbf{0},\\
&\mathbf{D}^{\mathrm{T}}\mathbf{a}_L + \mathbf{d}_R x_K + (\mathbf{b}_R + \mathbf{h}_R)x_D + \mathbf{f}_R x_M = \mathbf{0}.
\end{aligned} \tag{A.26}$$

The constraints on Class-2 elementary structure-preserving constraints have now been put in linear form as equations (A.22), (A.23) and (A.26).

APPENDIX B: CHANGES IN THE SYSTEM MATRICES AS A RESULT OF A CLASS-2 ELEMENTARY STRUCTURE-PRESERVING TRANSFORMATION

Let $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ represent the new system after a Class-2 elementary structure-preserving transformation has been applied.

The change occurring in the stiffness matrix is expressed initially as

$$\mathbf{K}' = (\mathbf{I} + \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{I} + \mathbf{a}_R\mathbf{b}_R^{\mathrm{T}}) - (\mathbf{f}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{M}(\mathbf{a}_R\mathbf{f}_R^{\mathrm{T}}). \tag{B.1}$$

Using the definitions of $\{x_K, x_D, x_M\}$ in the main text, this simplifies to

$$(\mathbf{K}' - \mathbf{K}) = \mathbf{b}_L\mathbf{a}_L^{\mathrm{T}}\mathbf{K} + \mathbf{K}\mathbf{a}_R\mathbf{b}_R^{\mathrm{T}} + \mathbf{b}_L\mathbf{b}_R^{\mathrm{T}}x_K - \mathbf{f}_L\mathbf{f}_R^{\mathrm{T}}x_M. \tag{B.2}$$

Using equations (16) of the main text to replace $\mathbf{K}\mathbf{a}_R$ and $\mathbf{a}_L^{\mathrm{T}}\mathbf{K}$ in equation (B.2) gives

$$\begin{aligned}(\mathbf{K} - \mathbf{K}') &= \mathbf{b}_L(\mathbf{b}_R^{\mathrm{T}}x_K + \mathbf{f}_R^{\mathrm{T}}x_D) + (\mathbf{b}_Lx_K + \mathbf{f}_Lx_D)\mathbf{b}_R^{\mathrm{T}} - \mathbf{b}_L\mathbf{b}_R^{\mathrm{T}}x_K + \mathbf{f}_L\mathbf{f}_R^{\mathrm{T}}x_M \\ &= \mathbf{b}_L\mathbf{b}_R^{\mathrm{T}}x_K + (\mathbf{b}_L\mathbf{f}_R^{\mathrm{T}} + \mathbf{f}_L\mathbf{b}_R^{\mathrm{T}})x_D + \mathbf{f}_L\mathbf{f}_R^{\mathrm{T}}x_M.\end{aligned} \tag{B.3}$$

The change occurring in the mass matrix is expressed initially as

$$\mathbf{M}' = (\mathbf{d}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{a}_R\mathbf{d}_R^{\mathrm{T}}) - (\mathbf{I} + \mathbf{h}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{M}(\mathbf{I} + \mathbf{a}_R\mathbf{h}_R^{\mathrm{T}}). \tag{B.4}$$

Using the definitions of $\{x_K, x_D, x_M\}$ in the main text, this simplifies to

$$(\mathbf{M}' - \mathbf{M}) = \mathbf{h}_L\mathbf{a}_L^{\mathrm{T}}\mathbf{M} + \mathbf{M}\mathbf{a}_R\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{h}_R^{\mathrm{T}}x_M - \mathbf{d}_L\mathbf{d}_R^{\mathrm{T}}x_K. \tag{B.5}$$

Using equations (16) of the main text to replace $\mathbf{M}\mathbf{a}_R$ and $\mathbf{a}_L^{\mathrm{T}}\mathbf{M}$ in equation (B.4) gives

$$\begin{aligned}(\mathbf{M} - \mathbf{M}') &= \mathbf{h}_L(\mathbf{d}_R^{\mathrm{T}}x_D + \mathbf{h}_R^{\mathrm{T}}x_M) + (\mathbf{d}_Lx_D + \mathbf{h}_Lx_M)\mathbf{h}_R^{\mathrm{T}} - \mathbf{h}_L\mathbf{h}_R^{\mathrm{T}}x_M + \mathbf{d}_L\mathbf{d}_R^{\mathrm{T}}x_K \\ &= \mathbf{d}_L\mathbf{d}_R^{\mathrm{T}}x_K + (\mathbf{d}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{d}_R^{\mathrm{T}})x_D + \mathbf{h}_L\mathbf{h}_R^{\mathrm{T}}x_M.\end{aligned} \tag{B.6}$$

The change occurring in the damping matrix can be expressed as

$$\mathbf{D}' = (\mathbf{d}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{I} + \mathbf{a}_R\mathbf{h}_R^{\mathrm{T}}) + (\mathbf{I} + \mathbf{h}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{K}(\mathbf{a}_R\mathbf{d}_R^{\mathrm{T}}) + (\mathbf{I} + \mathbf{h}_L\mathbf{a}_L^{\mathrm{T}})\mathbf{D}(\mathbf{I} + \mathbf{a}_R\mathbf{h}_R^{\mathrm{T}}). \tag{B.7}$$

Apply the definitions of $\{x_K, x_D, x_M\}$ in the main text

$$\begin{aligned}(\mathbf{D}' - \mathbf{D}) = {} & (\mathbf{d}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{d}_R^{\mathrm{T}})x_K + (\mathbf{d}_L\mathbf{a}_L^{\mathrm{T}}\mathbf{K} + \mathbf{K}\mathbf{a}_R\mathbf{d}_R^{\mathrm{T}}) + 2x_D\mathbf{h}_L\mathbf{h}_R^{\mathrm{T}} \\ & + (\mathbf{h}_L\mathbf{a}_L^{\mathrm{T}}\mathbf{D} + \mathbf{D}\mathbf{a}_R\mathbf{h}_R^{\mathrm{T}}).\end{aligned} \tag{B.8}$$

Using equations (16) of the main text to replace $\{\mathbf{K}\mathbf{a}_R, \mathbf{a}_L^{\mathrm{T}}\mathbf{K}, \mathbf{D}\mathbf{a}_R, \mathbf{a}_L^{\mathrm{T}}\mathbf{D}\}$ yields

$$\begin{aligned}(\mathbf{D'} - \mathbf{D}) = {} & (\mathbf{d}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{d}_R^{\mathrm{T}})x_K + 2x_D\mathbf{h}_L\mathbf{h}_R^{\mathrm{T}} \\ & - (\mathbf{d}_L(\mathbf{b}_R^{\mathrm{T}}x_K + \mathbf{f}_R^{\mathrm{T}}x_D) + (\mathbf{b}_Lx_K + \mathbf{f}_Lx_D)\mathbf{d}_R^{\mathrm{T}}) \\ & - (\mathbf{h}_L(\mathbf{d}_R^{\mathrm{T}}x_K + (\mathbf{b}_R^{\mathrm{T}} + \mathbf{h}_R^{\mathrm{T}})x_D + \mathbf{f}_R^{\mathrm{T}}x_M) + (\mathbf{d}_Lx_K + (\mathbf{b}_L + \mathbf{h}_L)x_D + \mathbf{f}_Lx_M)\mathbf{h}_R^{\mathrm{T}}),\end{aligned} \tag{B.9}$$

which simplifies to

$$\begin{aligned}(\mathbf{D} - \mathbf{D}') = {} & (\mathbf{b}_L\mathbf{d}_R^{\mathrm{T}} + \mathbf{d}_L\mathbf{b}_R^{\mathrm{T}})x_K \\ & + (\mathbf{d}_L\mathbf{f}_R^{\mathrm{T}} + \mathbf{f}_L\mathbf{d}_R^{\mathrm{T}} + \mathbf{b}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{b}_R^{\mathrm{T}})x_D + (\mathbf{f}_L\mathbf{h}_R^{\mathrm{T}} + \mathbf{h}_L\mathbf{f}_R^{\mathrm{T}})x_M.\end{aligned} \tag{B.10}$$

APPENDIX C: INVERSES OF CLASS-2 ELEMENTARY TRANSFORMATIONS

The structures of the general Class-2 elementary structure-preserving transformation (described by the five N vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{f}, \mathbf{h}\}$) and its inverse (described by the five N vectors $\{\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\}$) are implicitly given in equation (24) of the main text. Expansion of this yields

$$\mathbf{rs}^{\mathrm{T}} + \mathbf{ab}^{\mathrm{T}} + \mathbf{rs}^{\mathrm{T}}\mathbf{ab}^{\mathrm{T}} + \mathbf{rt}^{\mathrm{T}}\mathbf{af}^{\mathrm{T}} = \mathbf{0}, \tag{C.1}$$

$$\mathbf{ad}^{\mathrm{T}} + \mathbf{rs}^{\mathrm{T}}\mathbf{ad}^{\mathrm{T}} + \mathbf{rt}^{\mathrm{T}} + \mathbf{rt}^{\mathrm{T}}\mathbf{ah}^{\mathrm{T}} = \mathbf{0}, \tag{C.2}$$

$$\mathbf{ru}^{\mathrm{T}} + \mathbf{ru}^{\mathrm{T}}\mathbf{ab}^{\mathrm{T}} + \mathbf{af}^{\mathrm{T}} + \mathbf{rv}^{\mathrm{T}}\mathbf{af}^{\mathrm{T}} = \mathbf{0}, \tag{C.3}$$

$$\mathbf{ru}^{\mathrm{T}}\mathbf{ad}^{\mathrm{T}} + \mathbf{rv}^{\mathrm{T}} + \mathbf{ah}^{\mathrm{T}} + \mathbf{rv}^{\mathrm{T}}\mathbf{ah}^{\mathrm{T}} = \mathbf{0}. \tag{C.4}$$

Without loss of generality, it is possible to set

$$\mathbf{r} = \mathbf{a}. \tag{C.5}$$

From equation (C.1) to equation (C.5), the remaining unknown vectors $\{\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\}$ must be simple linear combinations of the known vectors as

$$\mathbf{s} = c_{sb}\mathbf{b} + c_{sf}\mathbf{f}, \qquad \mathbf{t} = c_{td}\mathbf{d} + c_{th}\mathbf{h}, \tag{C.6, C.7}$$

$$\mathbf{u} = c_{ub}\mathbf{b} + c_{uf}\mathbf{f}, \qquad \mathbf{v} = c_{vd}\mathbf{d} + c_{vh}\mathbf{h}. \tag{C.8, C.9}$$

Substitute for $\mathbf{s}$ and $\mathbf{u}$ in equations (C.1) and (C.3) using equations (C.6) and (C.8) and postmultiply the results by $\mathbf{b}$ and $\mathbf{f}$ in turn to obtain

$$\begin{bmatrix} (\mathbf{b}^{\mathrm{T}}\mathbf{b})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{b})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{b}^{\mathrm{T}}\mathbf{b})\,(\mathbf{d}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{b})\,(\mathbf{d}^{\mathrm{T}}\mathbf{a}) \\ (\mathbf{b}^{\mathrm{T}}\mathbf{f})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{f})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{b}^{\mathrm{T}}\mathbf{f})\,(\mathbf{d}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{f})\,(\mathbf{d}^{\mathrm{T}}\mathbf{a}) \\ (\mathbf{b}^{\mathrm{T}}\mathbf{b})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{b})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{b}^{\mathrm{T}}\mathbf{b})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{b})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) \\ (\mathbf{b}^{\mathrm{T}}\mathbf{f})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{f})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{b}^{\mathrm{T}}\mathbf{f})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) & (\mathbf{f}^{\mathrm{T}}\mathbf{f})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) \end{bmatrix} \begin{bmatrix} c_{sb} \\ c_{sf} \\ c_{ub} \\ c_{uf} \end{bmatrix} + \begin{bmatrix} (\mathbf{b}^{\mathrm{T}}\mathbf{b}) \\ (\mathbf{b}^{\mathrm{T}}\mathbf{f}) \\ (\mathbf{f}^{\mathrm{T}}\mathbf{b}) \\ (\mathbf{f}^{\mathrm{T}}\mathbf{f}) \end{bmatrix} = \mathbf{0}. \tag{C.10}$$

The rank of this system of equations, reduces from 4 to 2 if there is some scalar, ε, such that

$$\mathbf{f} = \mathbf{b}\varepsilon. \tag{C.11}$$

However, the equations do not become inconsistent. Neither is the solution for the inverse any less unique for this. In the case of equation (C.11), there is a one-dimensional space of acceptable choices for $\{c_{sb}, c_{sf}\}$ but every choice within this space produces the same $\mathbf{s}$ according to equation (C.6). Similarly in that case, there is a one-dimensional space of acceptable choices for $\{c_{ub}, c_{uf}\}$ but every choice within this space produces the same $\mathbf{u}$ according to equation (C.8). The equations become singular only when

$$(1 + \mathbf{b}^{\mathrm{T}}\mathbf{a})\,(1 + \mathbf{h}^{\mathrm{T}}\mathbf{a}) - (\mathbf{d}^{\mathrm{T}}\mathbf{a})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) = 0. \tag{C.12}$$

Substitute for **t** and **v** in equations (C.2) and (C.4) using equations (C.7) and (C.9) and postmultiply the results by **d** and **h** in turn to obtain

$$\begin{bmatrix} (\mathbf{d}^{\mathrm{T}}\mathbf{d})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{d})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{d}^{\mathrm{T}}\mathbf{d})\,(\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{d})\,(\mathbf{d}^{\mathrm{T}}\mathbf{a}) \\ (\mathbf{d}^{\mathrm{T}}\mathbf{h})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{h})\,(1+\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{d}^{\mathrm{T}}\mathbf{h})\,(\mathbf{b}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{h})\,(\mathbf{d}^{\mathrm{T}}\mathbf{a}) \\ (\mathbf{d}^{\mathrm{T}}\mathbf{d})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{d})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{d}^{\mathrm{T}}\mathbf{d})\,(1+\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{d})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) \\ (\mathbf{d}^{\mathrm{T}}\mathbf{h})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{h})\,(\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{d}^{\mathrm{T}}\mathbf{h})\,(1+\mathbf{f}^{\mathrm{T}}\mathbf{a}) & (\mathbf{h}^{\mathrm{T}}\mathbf{h})\,(1+\mathbf{h}^{\mathrm{T}}\mathbf{a}) \end{bmatrix} \begin{bmatrix} c_{sb} \\ c_{sf} \\ c_{ub} \\ c_{uf} \end{bmatrix} + \begin{bmatrix} (\mathbf{d}^{\mathrm{T}}\mathbf{d}) \\ (\mathbf{d}^{\mathrm{T}}\mathbf{h}) \\ (\mathbf{h}^{\mathrm{T}}\mathbf{d}) \\ (\mathbf{h}^{\mathrm{T}}\mathbf{h}) \end{bmatrix} = \mathbf{0}. \tag{C.13}$$

The rank of this system of equations reduces from 4 to 2 if there is some scalar, ε, such that

$$\mathbf{d} = \mathbf{h}\varepsilon. \tag{C.14}$$

Again, the equations do not become inconsistent—nor is the solution any less unique for this. In that case, there is a one-dimensional space of acceptable choices for $\{c_{td}, c_{th}\}$ but every choice within this space produces the same **t** according to equation (C.7). Similarly in that case, there is a one-dimensional space of acceptable choices for $\{c_{vd}, c_{vh}\}$ but every choice within this space produces the same **v** according to equation (C.9). The equations determining $\{c_{td}, c_{th}, c_{vd}, c_{vh}\}$ can become singular if and only if equation (C.12) applies.